Components:

$\{1, \dots, n\}$ are comp. or clique

if $\forall i,j \; i \to j$

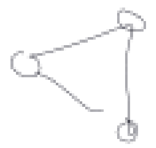
Giant?

specific

- $p = \frac{\log(n)}{n} + o\left(\frac{\log(n)}{n}\right)$ have giant $n - o(n)$
- $p = \frac{\log(n)}{n} - o\left(\frac{\log(n)}{n}\right)$ no giant, largest clique is $o(n)$

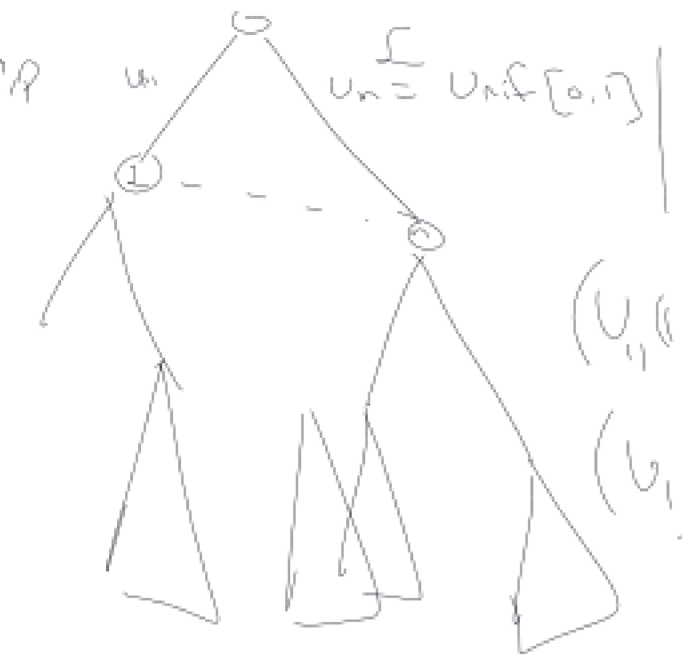
$p = \frac{c\log(n)}{n}$, $c < 1$,

? $\leq n$ largest clique is $\leq \left[1 + \frac{1}{1-c}\right]$



$p = \frac{1}{n}$

Uniform Temporal Trees



$T_{n,p}$ U_n $U_n \overset{\mathcal{L}}{=} \mathrm{Unif}[0,1]$

$\frac{|T_n|}{e^n} \xrightarrow{\mathcal{L}} \exp$

$\mathbb{E}\, T_n = \sum_{k=0}^{\infty} \frac{n^k}{k!} = e^n$

$(U_1, (1-U_1)U_2, \dots)$ - Subtree sizes

$(U_1, U_1U_2, U_1U_2U_3, \dots)$ - subtree sizes

- Profile: limit law for ~~$Z_k = \#$~~(vertices in gen k)?
  - Root recovery?

Random Simple Temporal Graph:

- $G_{n,p}$
- Uniform permutation on edges.

if there is an increasing path

regs in conn. happen around $\frac{c\log n}{n} = p$, $c \leq 4$

$U_1, \dots, U_n$

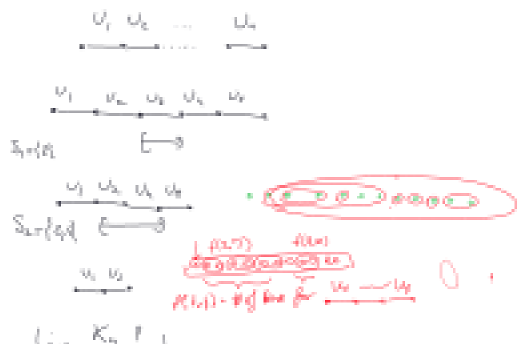
$K_n^1 = \min\left\{k : S_1 \cup \dots \cup S_k = [n], \forall i \sum_{j \in S_i} U_j \leq \frac{1}{\lambda}\right\}$

$\lim_{n\to\infty} \frac{K_n^1}{n} \overset{?}{=} \frac{1}{2}$

$K_n^2 = \min\{ \quad , S_i \text{ being int. of } [n]\}$

$\lim_{n\to\infty} \frac{K_n^2}{n} > \frac{1}{2}$

$K_n = \min\{ \quad , (S_1, \dots, S_k) \text{ line-connected}\}$



$\lim_{n\to\infty} \frac{K_n}{n} \overset{?}{=} \frac{1}{2}$

line-conn. case

general path case

H-conn. case the

Uniformizations

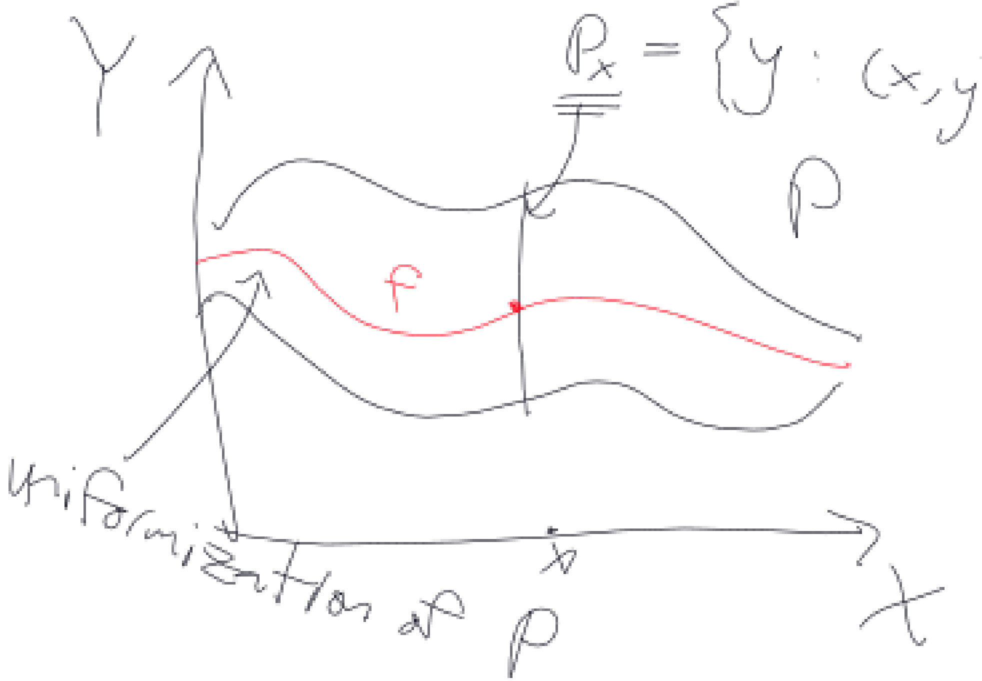
AC X, Y Polish spaces sets, $\{P_x\}_{x \in X}$ is a Borel family of non-empty subsets of Y, then $\exists$ Borel $f : X \to Y$ s.t. $f(x) \in P_x$ $\forall x \in X$.

Borel

$P \subseteq X \times Y$

$(x,y) \in P \iff y \in P_x$

$P_x = \{y : (x,y) \in P\}$



Fact There are some Borel P which do not admit Borel uniformizations

Standard results

$P_x = \{y : (x,y) \in P\}$

"large section"

Measure unif. If $\exists$ Borel prob. meas. μ on Y s.t. $\mu(P_x) > 0$ $\forall x$

Then $\exists$ Borel unif.

Category unif. If P_x is non-meagre $\forall x$, then $\exists$ Borel unif.

"small section"

K_σ unif (Arsenin, Kunugui) If P_x is K_σ $\forall x$, then $\exists$ Borel unif.

Ctbl unif (Lusin-Novikov) If P_x is ctbl $\forall x$, then $\exists$ Borel unif.

Invariant unif

Let E $\subseteq X \times X$ be a Borel equiv. relation on X.

Assume P is E-invariant, i.e., $x E x' \Rightarrow P_x = P_{x'}$

Q When is there a Borel E-inv. uniformization

i.e., a Borel unif $f: X \to Y$ s.t. $x E x' \Rightarrow f(x) = f(x')$

$A \subseteq X$

$P = X \times A \cap E$

Def^n E satisfies measure/category/K_σ/ctbl inv. unif

if whenever P is E-inv. & satisfies the hypotheses of the appropriate unif. thms, $\exists$ Borel E-inv. unif.

Thm TFAE:

(i) E is smooth

(ii) E sat. meas. inv. unif.

(iii) ——— cat. inv. unif

(iv) ——— K_σ ———

(v) ——— ctbl ———

let $A \subseteq 2^{\mathbb{N}}$ be E_0-inv. F_σ

$\rho = E_0$

& ~~...~~ meage μ-conull

μ product meas
unif

Def^n (1) E_0 is the equiv. rel. on $X = 2^{\mathbb{N}}$ of "eventual equality"

$x E_0 x' \iff \exists n \forall m \geq n \, (x(n) = x'(n))$

(2) E is smooth if $\exists$ Borel map $f: X \to Z$ (Polish) s.t. $x E x' \iff f(x) = f(x')$

$P_x = A + x$

If $f: 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ were a Borel E_0-inv. unif

$f \restriction C$ is const
comeage

$f(C) = y \implies C \subseteq A + y$

$\exists$ E not smooth, $\exists P$ with no E-inv. Borel unif. s.t.

(1) P_x is μ-conull $\forall x$, & P is $\underline{\underline{F_\sigma}}$ (some μ) (no meas. inv. unif)

(2) Q_x is comeagre $\forall x$, & Q is $\underline{\underline{G_\delta}}$ (no cat. inv. unif)

(3) R_x is ctbl $\forall x$, & $R \in \underline{\underline{F_\sigma}}$ (no ctbl K_σ inv. unif)

Thm $\forall E \forall P$

(ii) if $P_x \in F_\sigma$ & non-meagre $\forall x$, then $\exists$ Borel E-inv. unif

(iii) if $P_x \in G_\delta$ & K_σ $\forall x$, then $\exists$ Borel E-inv. unif

(i) if $P_x \in \underset{\sim}{\Delta}^0_2$ and $\mu(P_x) > 0$ $\forall x$, then $\exists$ Borel E-inv. unif

<u>Thm</u> let $X \subseteq 2^{\mathbb{N}}$ of seq. w/ ∞ly many 1^s

$\exists$ Polish space Y, a Borel prob. meas. μ on Y

and an E_0-inv. G_δ set $P \subseteq X \times Y$ st

$\mu(P_x) = 1$ & P_x is comeagre $\forall x$

but there is no E_0-inv. unif.

assume for a contra. that $\exists f$ Borel E_0-inv. unif.

Then $\exists B, A$ s.t. $f(C) = B$ $\forall C \in H$

let $H = [A]^{\aleph_0}$

$P(C, B)$ $\forall C \in H = [A]^{\aleph_0}$

$\neg P(A \cap B, B)$

$X = [\mathbb{N}]^{\aleph_0}$, $Y = 2^{\mathbb{N}}$

$P \subseteq X \times Y$, $P(A, B) \iff |A \setminus B| = |A \cap B| = \infty$

$\Updownarrow$

$(A, B) \in P$

P_A

Eg (1) Y be the space of graphs on $\mathbb{N}$.

$Q(A, G) \iff A$ contains "witnesses" that G is the rand. graph

$X = [\mathbb{N}]^{\aleph_0}$

$\in A$

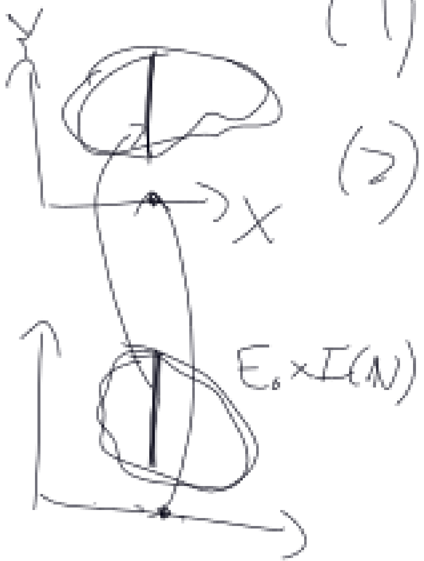
(2) $Y = [\mathbb{N}]^{\aleph_0}$

For $B \in Y$, let $f_B : \mathbb{N} \to \mathbb{N}$ be its increasing enum.

$R(A, B) \iff f_B(A)$ contains ∞-ly many even & odd elts.

Thm (Miller) Let P have ctbl sections. $\mathrm{Fix}(E,P)$ $P \subseteq X \times Y$

Exactly one of the following holds:



(1) $\exists$ Borel E-inv. unif.

(2) $\exists$ a cont. embedding $\pi_X : 2^{\mathbb{N}} \times \mathbb{N} \to X$ of $E_0 \times I(\mathbb{N})$ into E & a cont. inj. $\pi_Y : 2^{\mathbb{N}} \times \mathbb{N} \to Y$ s.t.

$$\neg (x \; E_0 \times I(\mathbb{N}) \; x') \Rightarrow P_{\pi_X(x)} \cap P_{\pi_X(x')} = \emptyset$$

$$P_{\pi_X(x)} = \{ \pi_Y(x') : x' \; E_0 \times I(\mathbb{N}) \; x \}$$

$\mathbb{N}^{\mathbb{N}}$ $\alpha \in \mathbb{N}^{\mathbb{N}}$ $X_\alpha \subseteq \mathbb{N}^{\mathbb{N}}$

$$X_\alpha = \{x \in \mathbb{N}^{\mathbb{N}} : x(n) \le \alpha(n) \ \infty\text{-ly often}\}$$

$S \subseteq \mathbb{N}^{<\mathbb{N}}$ $\forall u \in \mathbb{N}^{<\mathbb{N}}\ \exists s \in S$ s.t. $u \le s$

$T \subseteq \bigcup_n \mathbb{N}^n \times \mathbb{N}^n$ $\forall u, v \in$ $\exists (t_0, t_1) \in T$ s.t. $u \le t_0,\ v \le t_1$

$\forall n,\ |(S \cap \mathbb{N}^n) \cup (T \cap \mathbb{N}^n \times \mathbb{N}^n)| \le 1$

$$G_0^\omega = \{(s^\frown i^\frown x)_{i \in \mathbb{N}} : s \in S,\ x \in \omega^\omega\}$$

$$H_0^\omega = \{(t_0{}^\frown 0^\frown x,\ t_1{}^\frown 1^\frown x) : (t_0, t_1) \in T,\ x \in \omega^\omega\}$$

__Thm__ If G is a analytic $\aleph_0$-dim hypergraph on X ~ Polish and R is a analytic quasi-order on X, exactly one holds:

(1) $\exists$ a lin. order (quasi) $R' \supseteq R$ which is lexicographically red.

$(\exists$ Borel $f : X \to 2^{\alpha}, \alpha < \omega_1$, s.t. $x R' x' \Leftrightarrow f(x) \leq_{lex} f(x'))$

& a ctbl Borel $\equiv_R$-local coloring of G

$x \equiv_{R'} x' \Leftrightarrow x R' x'$ & $x' R' x$

(2) $\exists$ cont hom. $\varphi : X_\sigma \to X$ of $(G_0^\omega \restriction X_\sigma, H_0^\omega \restriction X_\sigma) \to (G, R)$